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# Systematic construction of multisoliton complexes

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We present a simple but powerful method for constructing multisoliton complexes of the coupled nonlinear Schrödinger equation. Our method is based on the Bäckrund-Darboux transformation. A closed form of the matrix determinant is given for multisoliton complexes, including the case of a nonvanishing background. We explicitly work out the solutions of two-, three-, and four-component coupled nonlinear Schrödinger equations.

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## I. INTRODUCTION

Recently, there has been much interest in multisoliton complexes (MSCs). A MSC is a self-localized state, which is a nonlinear superposition of several fundamental solitons. These objects have been studied extensively both experimentally and theoretically; see the review and references therein [1,2]. These include short pulses in multicore optical fibers [3], multicomponent Bose-Einstein condensates at zero temperature [4], and gap solitons [5], to name a few. In particular, researches on spatial incoherent solitons propagating in photorefractive materials induced new interest in MSCs [2]. In general, a MSC can be described by a set of coupled nonlinear Schrödinger equations (NLSEs). Various solutions to these equations, including soliton solutions [6,7] and periodic solutions, have been found [8–10], especially for the two-component case. Explicit solutions of equations having components larger than two are appearing one after another [2,8,11]. These solutions are mainly obtained using the fact that MSCs are stationary, which reduces the problem of the coupled NLSEs to a set of ordinary differential equations.

In some special cases, like wave propagation in a homogeneous medium having a Kerr-type nonlinear response, the corresponding NLSEs are found to be integrable. They are then described by

$$\partial_z \psi_k = -i\partial_x^2 \psi_k - 2i\sum_{i=0}^{N} |\psi_i|^2 \psi_k, \quad k=0,N. \tag{1}$$

The simplest case $N=1$ (two-component case) was known as the Manakov equation [12,13]. The inverse scattering method (ISM) [14] was used by Manakov for finding one soliton solution. The ISM is a powerful tool in constructing solitons, but the high-level mathematical technicality of the method makes it difficult for finding more complex solutions; multisolitons, and/or solitons, having nonvanishing backgrounds. Thus, most solutions of the MSC have been constructed in the form of stationary solutions, or using a linear superposition principle [15]. Some important results obtained in these ways are solitary waves solutions [16], MSC solutions of partially coherent solitons in Ref. [17], MSCs on a background [18,19], and MSCs in a sea of radiation modes [20]. Collisions of MSCs are also investigated and illustrated by numerical examples [21]. The stability of multicomponent solitary waves is studied in Ref. [22] and a Hirota bilinear method was used to find periodic solutions of coupled NLSEs [23,24].

These general analytical solutions were used to obtain some important characteristics of MSCs in a simple form. Due to its physical relevance, more explicit closed expressions of MSCs having a complex behavior should be required. In this paper, we present a simple, but powerful MSC finding technique that would serve well for finding more general solutions of MSCs. The method, which is based on the Darboux transformation (DT), uses the Crum's formula, and avoids the stationary ansatz [25–29]. When a DT is applied once in a given starting solution, it gives a new solution of (a soliton plus starting solution). To create a MSC on a starting solution $\psi^{(B)}$ (we restrict that a starting solution has a $\psi_0$ component only), we apply an $N$ iteration of the DT on $\psi^{(B)}$. In this course, we adjust that each added soliton, $i=1,N$, has two components only, $\psi_0$ and $\psi_i$. The final result is a closed determinantal form of MSC solutions in nonvanishing as well as vanishing backgrounds. Our method applies when the coupled NLSE is integrable. Clearly, the integrability from the Kerr-like nonlinearity is an approximation. Nevertheless, the existence of exact solutions helps us to understand the phenomenon. The general idea should be valid for any particular nonlinearity.

We explain the method in Sec. II and give a closed form of the MSC for $N=1,3$ equations in Eq. (1). We demonstrate the method for the completely incoherent MSCs in Sec. III. Partially coherent MSCs are discussed in Sec. IV. Incoherent MSCs on a background are explained in Sec. V. The validity of our formulas is verified independently by a computer algebra software. An explicit result is given in the Appendix which calculates MSCs, $\psi_i$, $i=0,2$, of an $N=2$ coupled NLSE on a background.

## II. METHOD

### A. Associated linear system for a MSC

We first bring the coupled NLS equation (1) into a matrix form in terms of $(N+2)\times(N+2)$ matrices $E,T$ and $\tilde{E}=[T,E]$,

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$$E=\begin{pmatrix} 0 & \psi_0 & \psi_1 & \cdots & \psi_N \\ -\psi_0^* & 0 & \cdots & & 0 \\ -\psi_1^* & 0 & \cdots & & 0 \\ \cdots & & & & \\ -\psi_N^* & 0 & \cdots & & 0 \end{pmatrix},$$

$$T=\begin{pmatrix} i/2 & 0 & \cdots & & 0 \\ 0 & -i/2 & 0 & \cdots & 0 \\ 0 & 0 & -i/2 & 0\cdots & 0 \\ \cdots & & & & \\ 0 & 0 & \cdots & 0 & -i/2 \end{pmatrix}, \tag{2}$$

such that

$$\partial_z E=-\partial_x^2\widetilde{E}+2E^2\widetilde{E}. \tag{3}$$

One can readily check that the components of Eq. (3) are indeed equivalent to the coupled NLS equation in Eq. (1) [12,28,29]. In this paper, we focus on the case when the group velocity dispersion is abnormal, or the waveguide is self-focusing. One advantage of using matrices is that we can write down the associated linear equation (Lax pair)

$$(\partial_x+E+\lambda T)\Psi=0,\quad (\partial_z+E\widetilde{E}-\partial_x\widetilde{E}-\lambda E-\lambda^2 T)\Psi=0, \tag{4}$$

where $\lambda$ is an arbitrary complex number and $\Psi(x,z,\lambda)$ is an $(N+1)$-component vector. What this linear equation means is the following: if the matrix $E$ satisfies Eq. (3), one can find a nonzero solution $\Psi$ by integrating Eq. (4). On the other hand, if there is a nontrivial $\Psi$ satisfying Eq. (4), then $\partial_x\partial_z\Psi$ should be the same as $\partial_z\partial_x\Psi$ for any value of $\lambda$, which when coupled with Eq. (4), requires that $E$ should satisfy Eq. (3).

With this prerequisite, we now introduce the Bäckrund-Darboux transformation, in a form suiting our purpose [26,28,29]. First, we choose a particular solution for $E$, $E=E^{(B)}$ with $\psi_0=\psi^{(B)},\psi_1=\cdots=\psi_N=0$, which later describes asymptotic backgrounds for the MSCs. For cases of completely incoherent MSCs, one may simply choose $\psi^{(B)}=0$. For cases of partially coherent (especially in $\psi_0$ component) MSCs, one can take $\psi^{(B)}=\psi_{(m)}$, where $\psi_{(m)}$ is an $m$-soliton solution of a single-component NLSE for $\psi_0$. For MSCs on a background, one may choose plane wave solutions for $\psi^{(B)}$ which can be easily obtained.

Now, with a choice of $E$ from the previous step, we integrate the linear equation (4) for pure imaginary cases, $\lambda_j=i\beta_j$, $(\beta_j=\text{real } j=1,N)$, in a form

$$\Psi(x,z,\lambda=i\beta_j)=\begin{pmatrix} a_j \\ b_j \\ 0 \\ \cdots \\ c_j \\ 0 \\ \cdots \end{pmatrix}. \tag{5}$$

Here $c_j$ lies in the $j+2$th row, which will add a soliton having a $\psi_j$ component (as well as a $\psi_0$ component through $b_j$) to a starting solution using the DT. In other words, it satisfies

$$\left\{\partial_x+\begin{pmatrix} 0 & \psi^{(B)} & 0 & \cdots & \\ -\psi^{(B)*} & 0 & \cdots & & \\ 0 & \cdots & & & \\ \cdots & & & & \\ \cdots & & & & \\ \cdots & & & & \\ 0 & \cdots & & & \end{pmatrix} + i\beta_j\begin{pmatrix} i/2 & 0 & \cdots & & 0 \\ 0 & -i/2 & 0 & \cdots & 0 \\ 0 & 0 & -i/2 & 0\cdots & 0 \\ \cdots & & & & \\ \cdots & & & & \\ \cdots & & & & \\ 0 & 0 & \cdots & 0 & -i/2 \end{pmatrix}\right\}\begin{pmatrix} a_j \\ b_j \\ 0 \\ \cdots \\ c_j \\ 0 \\ \cdots \end{pmatrix}=0, \tag{6}$$

and similarly for the $\partial_z$ part of Eq. (4). Here, we take $\lambda_j=i\beta_j$ to be purely imaginary, which makes MSCs stationary (or moving with a certain velocity) and pulsating periodically along their trajectories. These parameters $\beta_i$, $i=1,N$ are related to the intensities of MSC solitons.

### B. Crum's formula for MSCs

Now Crum's formula of the Darboux transformation gives a MSC solution in terms of $a_i, b_i, c_i$, $i=1,N$ described in Sec. II A as the solution of Eq. (4) or (6). A generalized form of Crum's formula for multi-component (Hermitian symmetric space) NLSE was first introduced in Ref. [28]. See more details in Ref. [29]. Define an $N(N+2)\times N(N+2)$ block matrix $D$, which is composed of $N^2$ block matrix $\Delta_{ij}$ such that

$$\mathbf{D}=\begin{pmatrix} \Delta_{1,1} & \Delta_{1,2} & \cdots & \Delta_{1,N} \\ \cdots & \cdots & & \cdots \\ \Delta_{N,1} & \Delta_{N,2} & \cdots & \Delta_{N,N} \end{pmatrix}, \tag{7}$$

where the $(N+2)\times(N+2)$ block matrices $\Delta_{mn}$ are given by the product of two block matrices;

$$\mathbf{\Delta_{mn}}=\begin{pmatrix} a_n & -b_n^* & 0 & \cdots & -c_n^* & 0 & \cdots & & \\ b_n & a_n^* & 0 & \cdots & & & & & \\ 0 & 0 & 1 & 0 & \cdots & & & & \\ 0 & 0 & 0 & 1 & 0 & \cdots & & & \\ \cdots & & & & & & & & \\ c_n & 0 & \cdots & & a_n^* & 0 & \cdots & & \\ 0 & \cdots & & & & 1 & 0 & \cdots & \\ \cdots & & & & & & & & \end{pmatrix}\begin{pmatrix} (i\beta_n)^{N-m} & 0 & 0 & \cdots \\ 0 & (-i\beta_n)^{N-m} & 0 & \cdots \\ 0 & 0 & (-i\beta_n)^{N-m} & \cdots \\ \cdots & \cdots & & \\ 0 & & & \end{pmatrix}. \tag{8}$$

The first matrix of the right part of Eq. (8) has four nondiagonal elements $b_n$, $b_n^*$, $c_n$, and $c_n^*$ which are located in the second row, the second column, the $n+2$nd row, and the $n+2$nd column, respectively. Explicitly, $D$ for $N=2$ becomes

$$\mathbf{D^{(N=2)}}=\begin{pmatrix} i\beta_1 a_1 & i\beta_1 b_1^* & i\beta_1 c_1^* & 0 & i\beta_2 a_2 & i\beta_2 b_2^* & 0 & i\beta_2 c_2^* \\ i\beta_1 b_1 & -i\beta_1 a_1^* & 0 & 0 & i\beta_2 b_2 & -i\beta_2 a_2^* & 0 & 0 \\ i\beta_1 c_1 & 0 & -i\beta_1 a_1^* & 0 & 0 & 0 & -i\beta_2 & 0 \\ 0 & 0 & 0 & -i\beta_1 & i\beta_2 c_2 & 0 & 0 & -i\beta_2 a_2^* \\ a_1 & -b_1^* & -c_1^* & 0 & a_2 & -b_2^* & 0 & -c_2^* \\ b_1 & a_1^* & 0 & 0 & b_2 & a_2^* & 0 & 0 \\ c_1 & 0 & a_1^* & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & c_2 & 0 & 0 & a_2^* \end{pmatrix}. \tag{9}$$

Let $Q_i$, $i=0,N$ be the matrix obtained by replacing the $i+1$st row of the matrix $D$ with the row matrix

$$[(i\beta_1)^N a_1 \quad -(-i\beta_1)^N b_1^* \quad -(-i\beta_1)^N c_1^* \quad 0 \quad \cdots \quad (i\beta_2)^N a_2 \quad -(-i\beta_2)^N b_2^* \quad 0 \quad -(-i\beta_2)^N c_2^* \quad \cdots]. \tag{10}$$

Explicitly, $Q_0$ and $Q_1$ for $N=2$ becomes

$$\mathbf{Q_0^{(N=2)}}=\begin{pmatrix} i\beta_1 a_1 & i\beta_1 b_1^* & i\beta_1 c_1^* & 0 & i\beta_2 a_2 & i\beta_2 b_2^* & 0 & i\beta_2 c_2^* \\ -\beta_1^2 a_1 & \beta_1^2 b_1^* & \beta_1^2 c_1^* & 0 & -\beta_2^2 a_2 & \beta_2^2 b_2^* & 0 & \beta_2^2 c_2^* \\ i\beta_1 c_1 & 0 & -i\beta_1 a_1^* & 0 & 0 & 0 & -i\beta_2 & 0 \\ 0 & 0 & 0 & -i\beta_1 & i\beta_2 c_2 & 0 & 0 & -i\beta_2 a_2^* \\ a_1 & -b_1^* & -c_1^* & 0 & a_2 & -b_2^* & 0 & -c_2^* \\ b_1 & a_1^* & 0 & 0 & b_2 & a_2^* & 0 & 0 \\ c_1 & 0 & a_1^* & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & c_2 & 0 & 0 & a_2^* \end{pmatrix}, \tag{11}$$

$$\mathbf{Q_1^{(N=2)}}=\begin{pmatrix} i\beta_1 a_1 & i\beta_1 b_1^* & i\beta_1 c_1^* & 0 & i\beta_2 a_2 & i\beta_2 b_2^* & 0 & i\beta_2 c_2^* \\ i\beta_1 b_1 & -i\beta_1 a_1^* & 0 & 0 & i\beta_2 b_2 & -i\beta_2 a_2^* & 0 & 0 \\ -\beta_1^2 a_1 & \beta_1^2 b_1^* & \beta_1^2 c_1^* & 0 & -\beta_2^2 a_2 & \beta_2^2 b_2^* & 0 & \beta_2^2 c_2^* \\ 0 & 0 & 0 & -i\beta_1 & i\beta_2 c_2 & 0 & 0 & -i\beta_2 a_2^* \\ a_1 & -b_1^* & -c_1^* & 0 & a_2 & -b_2^* & 0 & -c_2^* \\ b_1 & a_1^* & 0 & 0 & b_2 & a_2^* & 0 & 0 \\ c_1 & 0 & a_1^* & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & c_2 & 0 & 0 & a_2^* \end{pmatrix}. \tag{12}$$

Then a MSC solution for the coupled NLS equation is given by

$$\psi_0=\psi^{(B)}+i\frac{\det Q_0}{\det D},\quad \psi_i=i\frac{\det Q_i}{\det D},\quad i=1,N. \tag{13}$$

For a mathematical proof of these statements, we refer the reader to the reference [28] where the proof is given in a general context using the Darboux transformation and the generalized Crum's formula.

### C. Reduction of det $D$ and det $Q_i$

It can be seen that $D$ and $Q_i$, $i=0,N$ are large sparse matrices having many zeros. We find the determinants of them are reduced to more compact forms, which should be helpful for various applications. From now on, we take $\beta_1<\beta_2<\cdots<\beta_N$ without loss of generality. We first show that $\det D=|D|$ for $N=1,3$ is reduced to the following:

$$|D^{(N=1)}|=LP_1,$$

$$|D^{(N=2)}|=L\begin{vmatrix}-\beta_1M_1 & -\beta_2P_2\\ P_1 & M_2\end{vmatrix}+R\begin{Vmatrix}\alpha_1 & \alpha_2\\ \kappa_1 & \kappa_2\end{Vmatrix}^2,$$

$$|D^{(N=3)}|=L\begin{vmatrix}(-\beta_1)^2P_1 & (-\beta_2)^2M_2 & (-\beta_3)^2P_3\\ -\beta_1M_1 & -\beta_2P_2 & -\beta_3M_3\\ P_1 & M_2 & P_3\end{vmatrix}$$
$$+R(\beta_1^2-\beta_3^2)(\beta_2^2-\beta_3^2)\begin{Vmatrix}\alpha_1 & \alpha_2\\ \kappa_1 & \kappa_2\end{Vmatrix}^2\frac{P_3}{\beta_3}$$
$$+R(\beta_2^2-\beta_1^2)(\beta_3^2-\beta_1^2)\begin{Vmatrix}\alpha_2 & \alpha_3\\ \kappa_2 & \kappa_3\end{Vmatrix}^2\frac{P_1}{\beta_1}$$
$$+R(\beta_1^2-\beta_2^2)(\beta_3^2-\beta_2^2)\begin{Vmatrix}\alpha_1 & \alpha_3\\ \kappa_1 & \kappa_3\end{Vmatrix}^2\frac{M_2}{\beta_2}, \tag{14}$$

where

$$L=i^{N^2(N-1)/2}\prod_{i=1,N}\alpha_i^*\prod_{j=1,N,\ j>i}(\beta_j-\beta_i)^{N+1}(\beta_j+\beta_i)^3,$$

$$R=-4L\frac{\prod\beta_i}{\prod_{j<i}(\beta_j-\beta_i)}, \tag{15}$$

and $P_i=|\alpha_i|^2+|\kappa_i|^2+|\zeta_i|^2, M_i=|\alpha_i|^2+|\kappa_i|^2-|\zeta_i|^2, \alpha_i=a_i/l_i, \kappa_i=b_i/l_i, \zeta_i=c_i/m_i$,

$$l_i=\sqrt{\prod_{j=1,N\ j\neq i}(\beta_i+\beta_j)},\quad m_i=\sqrt{\prod_{j=1,N\ j\neq i}|\beta_i-\beta_j|}. \tag{16}$$

Here $\|\cdots\|^2$ means the squared absolute of a determinant. Similarly, $\det Q_i=|Q_i|$ are reduced to the following compact forms:

$$|Q_0^{(N=1)}|=2iL\beta_1a_1b_1^*,\quad |Q_1^{(N=1)}|=2iL\beta_1a_1c_1^*,$$

$$|Q_0^{(N=2)}|=-2iL\begin{vmatrix}\beta_1a_1b_1^* & \beta_2a_2b_2^*\\ P_1 & M_2\end{vmatrix},$$

$$|Q_i^{(N=2)}|=-2iL\begin{vmatrix}\delta_{i,1}\beta_1a_1c_1^* & \delta_{i,2}\beta_2a_2c_2^*\\ P_1 & M_2\end{vmatrix}+iRl_i\begin{vmatrix}\alpha_1 & \alpha_2\\ \kappa_1 & \kappa_2\end{vmatrix}\begin{vmatrix}\delta_{i,1}c_1^* & \delta_{i,2}c_2^*\\ \kappa_1^* & \kappa_2^*\end{vmatrix},\quad i=1,2,$$

$$|Q_0^{(N=3)}|=2iL\begin{vmatrix}\beta_1a_1b_1^* & \beta_2a_2b_2^* & \beta_3a_3b_3^*\\ -\beta_1M_1 & -\beta_2P_2 & -\beta_3M_3\\ P_1 & M_2 & P_3\end{vmatrix}+2iR\begin{vmatrix}\beta_1^2\alpha_1 & \beta_2^2\alpha_2 & \beta_3^2\alpha_3\\ \alpha_1 & \alpha_2 & \alpha_3\\ \kappa_1 & \kappa_2 & \kappa_3\end{vmatrix}\begin{vmatrix}\beta_1^2\kappa_1^* & \beta_2^2\kappa_2^* & \beta_3^2\kappa_3^*\\ \alpha_1^* & \alpha_2^* & \alpha_3^*\\ \kappa_1^* & \kappa_2^* & \kappa_3^*\end{vmatrix}, \tag{17}$$

$$|Q_i^{(N=3)}|=2iL\begin{vmatrix}\delta_{i,1}\beta_1 a_1 c_1^* & \delta_{i,2}\beta_2 a_2 c_2^* & \delta_{i,3}\beta_3 a_3 c_3^*\\ -\beta_1 M_1 & -\beta_2 P_2 & -\beta_3 M_3\\ P_1 & M_2 & P_3\end{vmatrix}+iRl_i\frac{(\beta_1^2-\beta_3^2)(\beta_2^2-\beta_3^2)}{(\beta_i+\beta_3)\beta_3}\begin{vmatrix}\alpha_1 & \alpha_2\\ \kappa_1 & \kappa_2\end{vmatrix}\begin{vmatrix}\delta_{i,1}c_1^* & \delta_{i,2}c_2^*\\ \kappa_1^* & \kappa_2^*\end{vmatrix}P_3$$

$$+iRl_i\frac{(\beta_2^2-\beta_1^2)(\beta_3^2-\beta_1^2)}{(\beta_i+\beta_1)\beta_1}\begin{vmatrix}\alpha_2 & \alpha_3\\ \kappa_2 & \kappa_3\end{vmatrix}\begin{vmatrix}\delta_{i,2}c_2^* & \delta_{i,3}c_3^*\\ \kappa_2^* & \kappa_3^*\end{vmatrix}P_1+iRl_i\frac{(\beta_1^2-\beta_2^2)(\beta_3^2-\beta_2^2)}{(\beta_i+\beta_2)\beta_2}\begin{vmatrix}\alpha_1 & \alpha_3\\ \kappa_1 & \kappa_3\end{vmatrix}\begin{vmatrix}\delta_{i,1}c_1^* & \delta_{i,3}c_3^*\\ \kappa_1^* & \kappa_3^*\end{vmatrix}M_2$$

$$+2iRl_i\begin{vmatrix}\beta_1^2\alpha_1 & \beta_2^2\alpha_2 & \beta_3^2\alpha_3\\ \alpha_1 & \alpha_2 & \alpha_3\\ \kappa_1 & \kappa_2 & \kappa_3\end{vmatrix}\begin{vmatrix}\delta_{i,1}c_1^* & \delta_{i,2}c_2^* & \delta_{i,3}c_3^*\\ \alpha_1^* & \alpha_2^* & \alpha_3^*\\ \kappa_1^* & \kappa_2^* & \kappa_3^*\end{vmatrix},\quad i=1,2,3.$$

Once again, MSC solutions are given by Eq. (13) and using these formulas. These formulas, Eqs. (14) and (17), are *our main result of present paper* and we discuss their applications to specific circumstances in the following sections. We check the correctness of these formulas using the symbolic package MAPLE. Another package, MATHEMATICA, was used to draw various figures for $\psi_i$ in the following sections, as well as to check that $\psi_i$ indeed satisfy the coupled NLSE (1).

## III. COMPLETELY INCOHERENT MSC

Completely incoherent MSCs can be obtained by choosing a trivial solution $\psi^{(B)}=0$ in the linear equation (6), and integrating it with a result

$$\Psi(x,z,\lambda=i\beta_j)=\begin{pmatrix}a_j=l_j\alpha_j=l_j\exp(\beta_j X_j),\\ b_j=l_j\kappa_j=0\\ 0\\ \cdots\\ c_j=m_j\zeta_j=m_j\exp(-\beta_j X_j)\\ 0\\ \cdots\end{pmatrix},\quad X_j=(x-i\beta_j z)/2. \tag{18}$$

Here we omit constants of integration, for simplicity. (Their effect can be incorporated by taking $x\rightarrow x-x_j, z\rightarrow z-z_j$.) Note that we take $b_j=\kappa_j=0$ in this case. It is now easy to obtain $N\times N$ matrices $D$ and $Q_i$ [omitting the $L$ factor in Eq. (14)]:

$$|D|\propto\begin{vmatrix}2\cosh\beta_1 x & 2\sinh\beta_2 x & 2\cosh\beta_3 x & \cdots\\ 2(-\beta_1)\sinh\beta_1 x & 2(-\beta_2)\cosh\beta_2 x & 2(-\beta_3)\sinh\beta_3 x & \cdots\\ 2(-\beta_1)^2\cosh\beta_1 x & 2(-\beta_2)^2\sinh\beta_2 x & 2(-\beta_3)^2\cosh\beta_3 x & \cdots\\ \cdots & & & \\ 2(-\beta_1)^{N-1}\cosh(\text{or}\sinh) & \cdots & & \end{vmatrix} \tag{19}$$

and

$$|Q_i|\propto 2i(-)^{[N(N-1)]/2}\begin{vmatrix}2\cosh\beta_1 x & 2\sinh\beta_2 x & 2\cosh\beta_3 x & \cdots\\ 2(-\beta_1)\sinh\beta_1 x & 2(-\beta_2)\cosh\beta_2 x & 2(-\beta_3)\sinh\beta_3 x & \cdots\\ 2(-\beta_1)^2\cosh\beta_1 x & 2(-\beta_2)^2\sinh\beta_2 x & 2(-\beta_3)^2\cosh\beta_3 x & \cdots\\ \cdots & & & \\ \delta_{i,1}\beta_1 a_1 c_1^* & \delta_{i,2}\beta_2 a_2 c_2^* & \delta_{i,3}\beta_3 a_3 c_3^* & \cdots\end{vmatrix},\quad i=1,N. \tag{20}$$

In this case, $Q_0=0$ and

$$\psi_0=0,\quad \psi_i=i\frac{|Q_i|}{|D|},\quad i=1,N. \tag{21}$$

These are completely incoherent MSCs of the $N$-coupled NLS system. Note that $Q_i$ is different from $D$ only in the last row. More explicitly, $\psi_0^{(N=1)}=0$, $\psi_1^{(N=1)}=-\beta_1\,\mathrm{sech}\,\beta_1 x\exp(-i\beta_1^2 z)$ for $N=1$ case. The results of

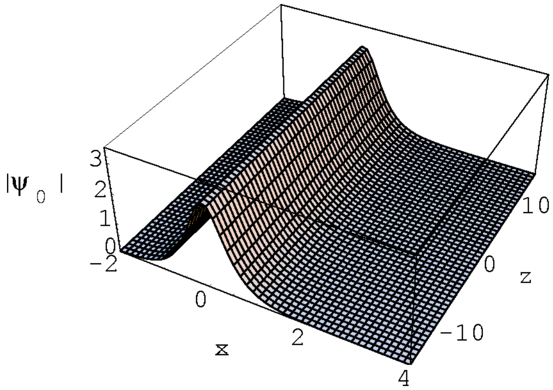


FIG. 1. Starting one-soliton background $\psi^{(B)}$ with $\beta_0=2.8$.

higher $N$ ($N>1$) cases can be expressed in more compact forms using identities of hyperbolic trigonometric functions. For example, $|D^{(N=2)}|=2(\beta_1+\beta_2)\cosh(\beta_1-\beta_2)x+2(\beta_2-\beta_1)\cosh(\beta_1+\beta_2)x$. These results are well known from previous researches [2,21]. These types of multisoliton solutions expressed as the ratio of two determinants were famous in the world of single-component NLSEs [26].

### IV. PARTIALLY COHERENT MSC

These cases arise when the background $\psi^{(B)}$ is given as an $M$ soliton of single-component NLSE. It then describes a case of $M+N$ solitons, interacting coherently through the $\psi_0$ component. Here we consider the $M=1$ case, for simplicity. For a one-soliton background, we take $\psi^{(B)}=-\beta_0\,\mathrm{sech}(\beta_0 x)\exp(-i\beta_0^2 z)$. Then the integration of linear equations (4) gives $\Psi(x,z,\lambda=i\beta_j)$ in Eq. (5) with

$$
\begin{aligned}
a_j&=il_j(\beta_0+\beta_j)s_j\exp\beta_jX_j-il_j\beta_0\,\mathrm{sech}\,\beta_0x\\
&\quad\times[s_j\exp(\beta_0x+\beta_jX_j)+r_j\exp(-i\beta_0^2z-\beta_jX_j)],\\
b_j&=il_j(\beta_0+\beta_j)r_j\exp(-\beta_jX_j)-il_j\beta_0\,\mathrm{sech}\,\beta_0x\\
&\quad\times[s_j\exp(i\beta_0^2z+\beta_jX_j)+r_j\exp(-\beta_0x-\beta_jX_j)],\\
c_j&=m_j\exp(-\beta_jX_j+t_j),
\end{aligned}
\tag{22}
$$

where $s_j$, $r_j$, and $t_j$ are arbitrary constants and $l_j$, $m_j$, and $X_j$ are defined in Eqs. (16) and (18). Explicit construction of soliton complexes can be done using Eqs. (13), (14), and (17). Though the expression looks still complicated, one can evaluate the $N$-soliton complexes with the help of the computer algebra system. Figure 1 shows a starting solution $\psi^{(B)}$, which is the well-known one-soliton of sech type. Figure 2 shows results of $N=1$ formulas in Eqs. (13), (14), and (17) with parameters $\beta_0=2.8, \beta_1=2.9, r_1=1, s_1=0$, and $t_1=0$. These figures show characteristic solitons of two bright pair. In this case, there is no oscillating behavior. Figure 3 shows results of $N=1$ calculation with the same parameters as in Fig. 2 except $s_1=1$. A noticeable difference between Figs. 2 and 3 is the oscillating behavior in Fig. 3. This is due to the interference of two terms in Eq. (22) (one is coupled to $r_1$ and the other is coupled to $s_1$.) of the linear equations. Generally, an oscillating behavior appears when two terms, each coupled to $r_i$ and $s_i$, are used to construct the soliton complexes. It is impossible to find these oscillating behaviors using the method of stationary ansatz. Figure 4 show results for the $N=2$ case using parameters of $r_i=s_i=1$ and $i=1,2$. It also shows the oscillating behavior. Solitons in these partially coherent MSCs interact with each other coherently through the $\psi_0$ component. Finally, Fig. 5 shows the MSC of the $N=3$ case. It has four $\psi_i$, $i=0,3$ components. This would be the most complex MSCs found upto now.

### V. MSCs ON A CONTINUOUS WAVE BACKGROUND

A dark soliton arises as a localized dip in a continuous wave (cw) background, $\psi^{(B)}=\beta_0\exp(-i\beta_0^2z/2)/2$. It was known that the cw background has intrinsic instabilities [30], and the physical application of solutions in this section is limited to special circumstances, see, for example, Ref. [31]. An explicit integration of the linear equation (4) with this $\psi^{(B)}$ gives $\Psi(x,z,\lambda=i\beta_j)$ in Eq. (5) with

$$
\begin{aligned}
a_j&=l_j\exp(-i\beta_0^2z/4)\\
&\quad\times\left\{\frac{s_j}{M_{j-}}\exp\left(\sqrt{\beta_j^2-\beta_0^2}X_j\right)+\frac{r_j}{M_{j+}}\exp\left(-\sqrt{\beta_j^2-\beta_0^2}X_j\right)\right\},\\
b_j&=l_j\exp(i\beta_0^2z/4)\{s_jM_{j-}\exp(\sqrt{\beta_j^2-\beta_0^2}X_j)\\
&\quad+r_jM_{j+}\exp(-\sqrt{\beta_j^2-\beta_0^2}X_j)\},\\
c_j&=m_j\exp(-\beta_jX_j+t_j),
\end{aligned}
\tag{23}
$$

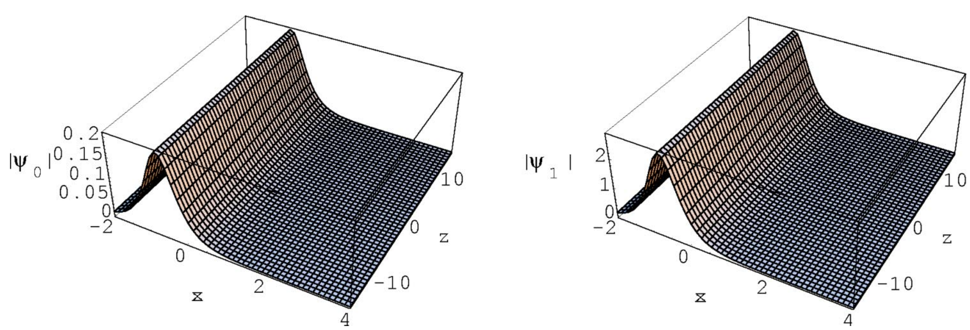


FIG. 2. Two-component MSC that is constructed by adding one soliton on a soliton background. Parameters are $\beta_0=2.8$, $\beta_1=2.9$, $r_1=1$, $s_1=0$, and $t_1=0$.

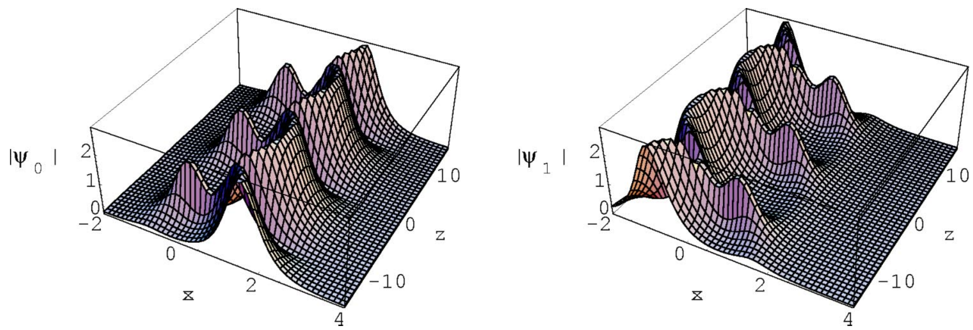


FIG. 3. Two-component MSC showing an oscillating behavior. Parameters are $\beta_0=2.8$, $\beta_1=2.9$, $r_1=1$, $s_1=1$, and $t_1=0$.

where $s_j, r_j$, and $t_j$ are arbitrary constants and $M_{j\pm}=(\sqrt{\beta_j+\beta_0}\pm\sqrt{\beta_j-\beta_0})/\sqrt{2\beta_0}$. With these elementary solutions, the rest of the step is just the same as in the previous sections.

Here, we give explicit expressions for $\psi_0,\psi_j$, $j=1,N$ for $N=1$ case, which are obtained using Eqs. (13), (14), (17), and (23). Expressions for $\psi_0,\psi_j$, $j=1,N$ for $N=2$ case are given in the Appendix. For the $N=1$ case (for simplicity, we take $t_1=0$)

$$\begin{aligned}\psi_0=&\frac{1}{2U}\exp(-i\beta_0^2 z/2)\{\beta_0^2\exp(-\beta_1 x)-2r_1^2\beta_0\beta_1\\&\times\exp(-\sqrt{\beta_1^2-\beta_0^2}x)-2s_1^2\beta_0\beta_1\exp(\sqrt{\beta_1^2-\beta_0^2}x)\\&+4r_1s_1[2i\beta_1\sqrt{\beta_1^2-\beta_0^2}\sin(\beta_1\sqrt{\beta_1^2-\beta_0^2}z)\\&-(2\beta_1^2-\beta_0^2)\cos(\beta_1\sqrt{\beta_1^2-\beta_0^2}z)]\},\end{aligned}$$

$$\begin{aligned}\psi_1=&-\frac{\sqrt{2\beta_0}\beta_1}{U}\exp[-\beta_1 x/2-i(\beta_0^2+2\beta_1^2)z/4]\\&\times\{s_1(\sqrt{\beta_1+\beta_0}+\sqrt{\beta_1-\beta_0})\exp(\sqrt{\beta_1^2-\beta_0^2}X_1)\\&+r_1(\sqrt{\beta_1+\beta_0}-\sqrt{\beta_1-\beta_0})\exp(-\sqrt{\beta_1^2-\beta_0^2}X_1)\},\end{aligned}\tag{24}$$

$$\begin{aligned}U=&\beta_0\exp(-\beta_1 x)+2r_1^2\beta_1\exp(-\sqrt{\beta_1^2-\beta_0^2}x)\\&+2s_1^2\beta_1\exp(\sqrt{\beta_1^2-\beta_0^2}x)+4\beta_0r_1s_1\cos(\beta_1\sqrt{\beta_1^2-\beta_0^2}z).\end{aligned}$$

The intensity profiles experience a periodic beating due to the interference between terms coupled to $r_1$ and $s_1$, respectively. A stationary solution is obtained by taking $s_1=1, r_1=0$, for instance. Then they become the famous dark-bright pair of soliton solution [8],

$$\psi_0=-\frac{\beta_0}{2}\exp(-i\beta_0^2 z/2)\tanh\left(\frac{\beta_1+\sqrt{\beta_1^2-\beta_0^2}}{2}(x+x_0)\right),$$

$$\begin{aligned}\psi_1=&-\frac{\sqrt{\beta_1}}{2}(\sqrt{\beta_1+\beta_0}+\sqrt{\beta_1-\beta_0})\exp[-i(\beta_0^2+2\beta_1^2\\&+2\beta_1\sqrt{\beta_1^2-\beta_0^2})z/4]\mathrm{sech}\left(\frac{\beta_1+\sqrt{\beta_1^2-\beta_0^2}}{2}(x+x_0)\right),\end{aligned}\tag{25}$$

where a constant $x_0$ satisfies

$$\exp\left(\frac{\beta_1+\sqrt{\beta_1^2-\beta_0^2}}{2}x_0\right)=\sqrt{2\beta_1/\beta_0}.$$

We draw some MSCs on a cw background with the help of symbolic package MATHEMATICA. Figure 6 shows $N=1$ MSC on a cw background with parameters $\beta_0=2.8$, $\beta_1=2.9$, $r_1=1$, $s_1=0$, and $t_1=0$. These figures are those of characteristic dark-bright pair. Figure 7 shows the results of $N=1$ MSC with the same parameters as in Fig. 6 except $s_1=1$. As expected, there is an oscillating behavior. We note

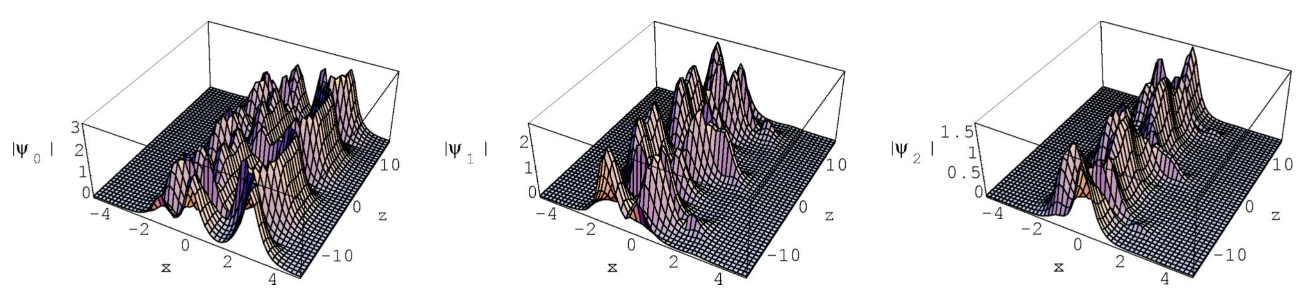


FIG. 4. Three-component MSC which is "Darboux constructed" by adding two solitons on a soliton background. Parameters are $\beta_0=2.8$, $\beta_1=2.9$, $\beta_2=3.1$, $r_i=s_i=1$, $t_i=0$, and $i=1,2$.

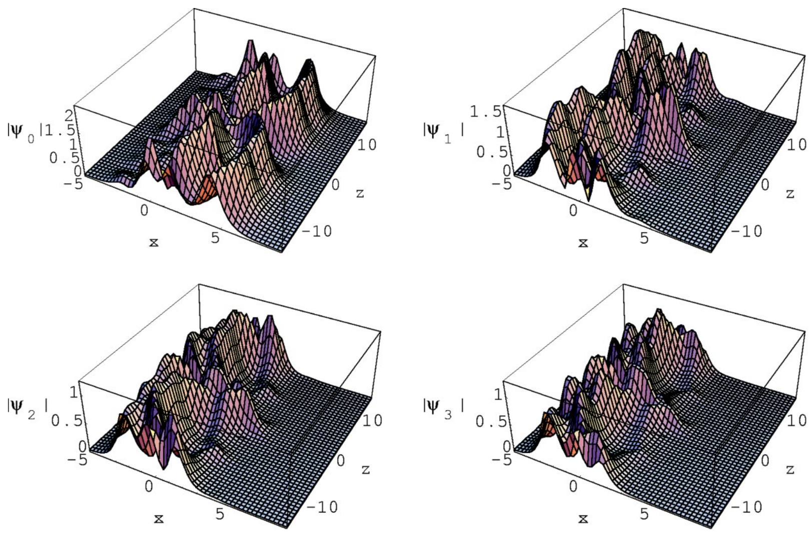


FIG. 5. Four-component MSC which is Darboux constructed by adding three solitons on a soliton background. Parameters are $\beta_0=1.8$, $\beta_1=1.9$, $\beta_2=2.1$, $\beta_3=2.2 r_i=s_i=1$, and $t_i=0$, $i=1,2$.

that it resembles the breather solution of single-component NLSE. Figure 8 shows $N=2$ MSC on a cw background, having parameters $r_i=s_i=1$, and $i=1,2$. It also shows the oscillating behavior. Finally, Fig. 9 shows an $N=3$ MSC on a cw background, having four $\psi_i$, $i=0,3$ components.

## VI. CONCLUSION

In this paper, we use a Darboux transformation for constructing MSC solutions of coupled NLSEs. A few explicit matrix determinants of small size the ($N=1,2,3$ case) are constructed from the large sparse matrices of the Crum's formula. Then MSCs are obtained by taking the ratio of two newly obtained matrix determinants. Our method not only reproduces known MSCs, but it can produce many new type of MSCs. For example, MSCs having a pulsating behavior are constructed. Formulas of $N \geqslant 4$ could be conjectured, but the proof of them needs to calculate the determinant of $N(N+2)\times N(N+2)$ block matrices, which has not been carried out even with the help of symbolic packages. For the case of complex $\beta_i$, $i=1,N$, there should appear more complex MSC solutions having soliton fusion or breakup phenomena [32].

Our method can be easily generalized to the defocusing case. In this case, some minus signs in Eq. (13), as well as in $b_i$ and $c_i$ in Eq. (5) produce correct MSCs of defocusing NLSE. More detailed results will be reported elsewhere. Our method also works for more general background other than the continuous wave background. An important case belonging to this is the cnoidal wave background [33], which appears to have interesting applications [34]. Though we have focused only on the generalized Manakov system in this paper, our method also applies to several other systems like the multicomponent self-induced transparency equation (SIT) which shares the Darboux covariance property [35].


## ACKNOWLEDGMENT

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## APPENDIX: $N=2$ MSC ON A BACKGROUND

Substituting $a_i, b_i, c_i$, $i=1,2$ in Eq. (23) into $N=2$ case of Eqs. (14) and (17), we obtain

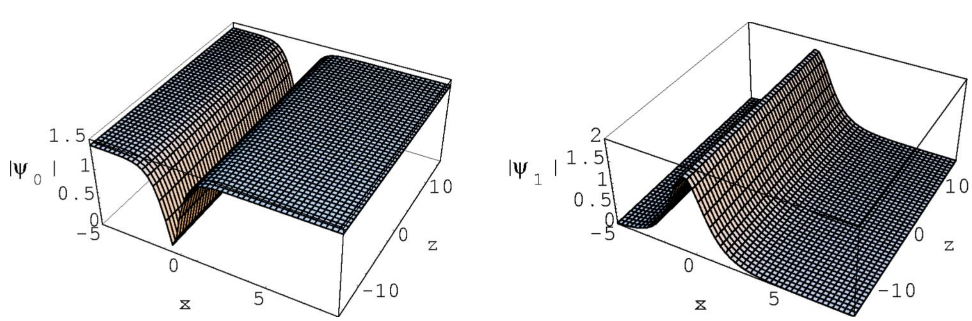


FIG. 6. $N=1$ MSC on a cw background. Parameters are $\beta_0=2.8$, $\beta_1=2.9$, $r_1=1$, $s_1=0$, and $t_1=0$.

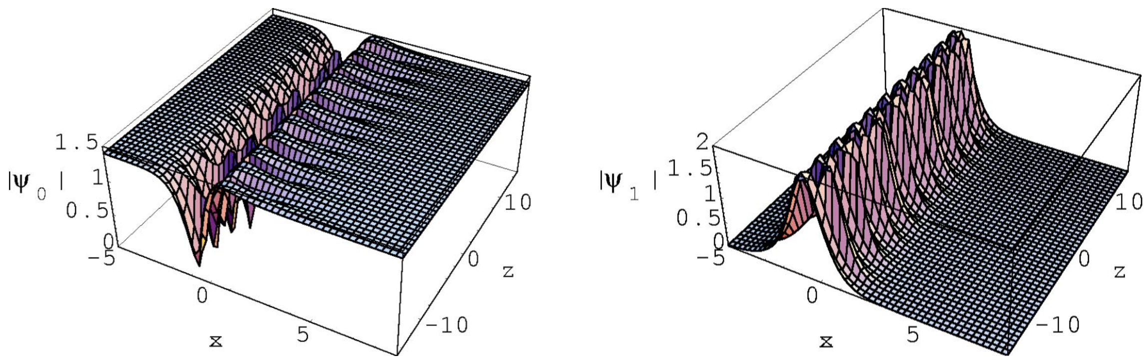

FIG. 7. $N=1$ MSC on a cw background. Parameters are $\beta_0=2.8$, $\beta_1=2.9$, $r_1=1$, $s_1=1$, and $t_1=0$.

$$
\begin{aligned}
|D|=&-\frac{1}{\beta_0^2(\beta_2-\beta_1)}\{-16\beta_1\beta_2(\beta_1^2+\beta_2^2-2\beta_0^2)C_1C_2+32\beta_1\beta_2\sqrt{\beta_1^2-\beta_0^2}\sqrt{\beta_2^2-\beta_0^2}S_1S_2-\beta_0^2(\beta_2-\beta_1)^2E_1E_2\\
&-2\beta_0(\beta_2^2-\beta_1^2)(2\beta_1(E_2-4O_2)C_1+2\beta_2(E_1+4O_1)C_2)-4\beta_0^2(\beta_2^2-\beta_1^2)(E_2O_1+E_1O_2)\\
&-16(\beta_2^2\beta_0^2+\beta_1^2\beta_0^2-2\beta_1^2\beta_2^2)O_1O_2+32\beta_1\beta_2\sqrt{\beta_1^2-\beta_0^2}\sqrt{\beta_2^2-\beta_0^2}I_1I_2\},
\end{aligned}
$$

$$
\begin{aligned}
|Q_0|=&2i\frac{\beta_1+\beta_2}{\beta_0^2}\exp(-i\beta_0^2z/2)\{8i\beta_0(\beta_1\sqrt{\beta_1^2-\beta_0^2}O_2I_1-\beta_2\sqrt{\beta_2^2-\beta_0^2}O_1I_2)+8\beta_0(\beta_2^2-\beta_1^2)O_1O_2\\
&+8\beta_2[(\beta_0^2-\beta_1^2)O_1+i\beta_1\sqrt{\beta_1^2-\beta_0^2}I_1]C_2-8\beta_1[(\beta_0^2-\beta_2^2)O_2+i\beta_2\sqrt{\beta_2^2-\beta_0^2}I_2]C_1\\
&+2\beta_2\beta_0(\beta_2O_2-i\sqrt{\beta_2^2-\beta_0^2}I_2)E_1+2\beta_1\beta_0(\beta_1O_1-i\sqrt{\beta_1^2-\beta_0^2}I_1)E_2+2\beta_2\beta_0^2E_1C_2+2\beta_1\beta_0^2E_2C_1\},
\end{aligned}
\tag{A1}
$$

$$
\begin{aligned}
|Q_1|=&i\sqrt{\frac{2}{\beta_0^3}}\beta_1\sqrt{\frac{\beta_2+\beta_1}{\beta_2-\beta_1}}\exp(-\beta_1x/2+t_1-i\beta_1^2z/2-i\beta_0^2z/4)\{(\sqrt{\beta_1+\beta_0}H_1+\sqrt{\beta_1-\beta_0}J_1)[8\beta_0\beta_1O_2+2\beta_0(\beta_2-\beta_1)E_2\\
&-8\beta_2\sqrt{\beta_2^2-\beta_0^2}S_2+8\beta_2\beta_1C_2]+(\sqrt{\beta_1+\beta_0}H_1-\sqrt{\beta_1-\beta_0}J_1)(-8\beta_2^2O_2+8i\beta_2\sqrt{\beta_2^2-\beta_0^2}I_2-8\beta_2\beta_0C_2)\},
\end{aligned}
$$

where

$$C_i=r_is_i\cosh\sqrt{\beta_i^2-\beta_0^2}(x+u_i),$$

$$S_i=r_is_i\sinh\sqrt{\beta_i^2-\beta_0^2}(x+u_i),$$

$$E_i=\exp(-\beta_ix+2t_i),\quad O_i=r_is_i\cos\beta_i\sqrt{\beta_i^2-\beta_0^2}z,$$

$$I_i=r_is_i\sin\beta_i\sqrt{\beta_i^2-\beta_0^2}z,$$

$$H_i=s_1\exp(\sqrt{\beta_i^2-\beta_0^2}X_i)/2+r_1\exp(-\sqrt{\beta_i^2-\beta_0^2}X_i)/2,\tag{A2}$$

$$J_i=s_1\exp(\sqrt{\beta_i^2-\beta_0^2}X_i)/2-r_1\exp(-\sqrt{\beta_i^2-\beta_0^2}X_i)/2.$$

Here $u_i=\ln(s_i/r_i)/\sqrt{\beta_i^2-\beta_0^2}$, $i=1,2$. $|Q_2|$ is obtained from the expression for $|Q_1|$ by exchanging $1\leftrightarrow2$. Finally, $\psi_i$, $i=0,2$ are obtained by substituting previous formulas into Eq. (13).

When we take $r_1=r_2=0$ (no pulsating behavior) and

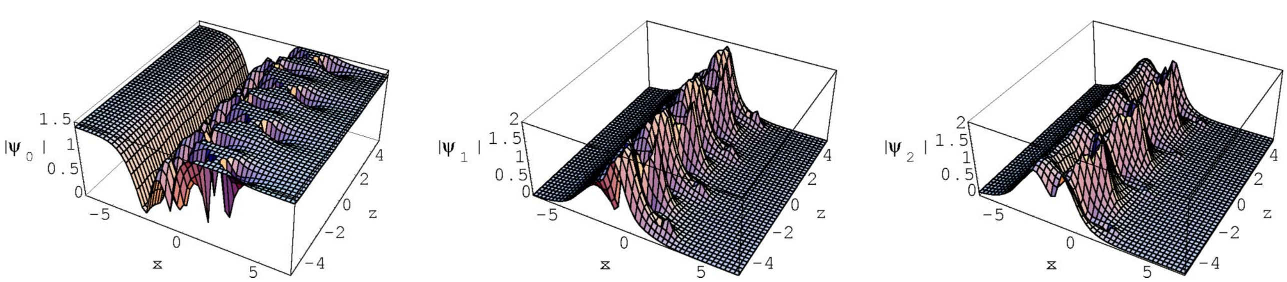

FIG. 8. $N=2$ MSC on a cw background. Parameters are $\beta_0=2.8$, $\beta_1=2.9$, $\beta_2=3.1$, $r_i=s_i=1$, $t_i=0$, and $i=1,2$.

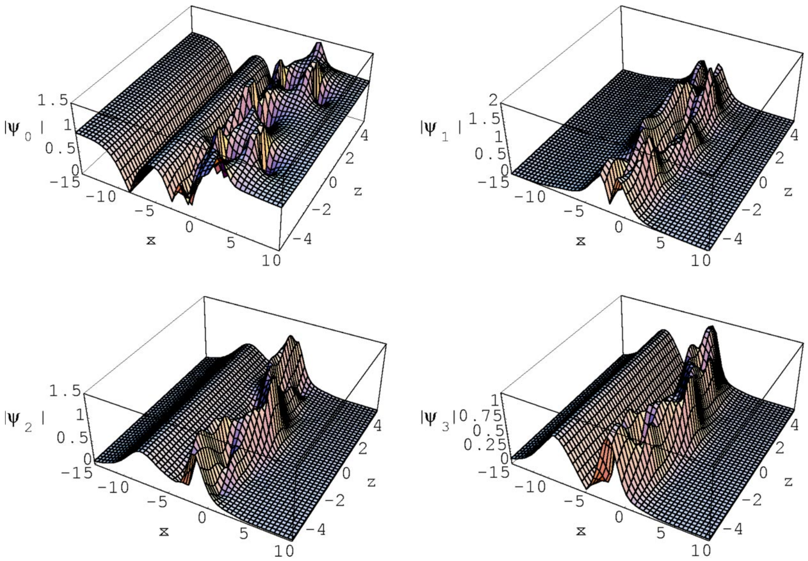


FIG. 9. $N=3$ MSC on a cw background. Parameters are $\beta_0=1.8$, $\beta_1=1.9$, $\beta_2=2.1$, $\beta_3=2.2 r_i=s_i=1$, $t_i=0$, and $i=1,2$.

$$s_i=\frac{\sqrt{6}}{2}\sqrt{\frac{\beta_0(\beta_2-\beta_1)}{\beta_i(\beta_2+\beta_1)}},\quad i=1,2 \tag{A3}$$

and

$$\beta_2+\sqrt{\beta_2^2-\beta_0^2}=2(\beta_1+\sqrt{\beta_1^2-\beta_0^2})$$

$$\text{i.e.,}\quad \beta_2=\frac{8\beta_1^2-3\beta_0^2+8\beta_1\sqrt{\beta_1^2-\beta_0^2}}{4(\beta_1+\sqrt{\beta_1^2-\beta_0^2})}, \tag{A4}$$

the above system reduces to

$$|D|=8(\beta_2-\beta_1)A\cosh^3\Delta x,$$

$$|Q_0|=6i\beta_0(\beta_2-\beta_1)\exp(-i\beta_0^2 z/2)A\cosh\Delta x,$$

$$|Q_1|=-4i\sqrt{3\beta_1}(\sqrt{\beta_1-\beta_0}+\sqrt{\beta_1+\beta_0})(\beta_2-\beta_1)\times\exp(-i\Delta^2 z-i\beta_0^2 z/2)A\cosh\Delta x\sinh\Delta x, \tag{A5}$$

$$|Q_2|=2i\sqrt{3\beta_2}(\sqrt{\beta_2-\beta_0}+\sqrt{\beta_2+\beta_0})(\beta_2-\beta_1)\times\exp(-4i\Delta^2 z-i\beta_0^2 z/2)A\cosh\Delta x,$$

where $A=\exp((-\beta_1+\sqrt{\beta_1^2-\beta_0^2}-\beta_2+\sqrt{\beta_2^2-\beta_0^2})x/2)$ and $\Delta=(\beta_1+\sqrt{\beta_1^2-\beta_0^2})/2$. With these results, we can obtain a $N=2$ MSC on a continuous wave background, which has two independent parameters $\beta_0$ and $\beta_1$ as following [2,18]. [$\beta_2$ in the expression should be substituted by Eq. (A4).]

$$\psi_0=\psi_{cw}+i\frac{|Q_0|}{|D|}=-\frac{1}{4}\exp(-i\beta_0^2 z/2)\beta_0(1-3\tanh^2\Delta x),$$

$$\psi_1=i\frac{|Q_1|}{|D|}=\frac{1}{2}\exp(-i\Delta^2 z-i\beta_0^2 z/2)\times\sqrt{3\beta_1}(\sqrt{\beta_1-\beta_0}+\sqrt{\beta_1+\beta_0})\operatorname{sech}\Delta x\tanh\Delta x, \tag{A6}$$

$$\psi_2=i\frac{|Q_2|}{|D|}=-\frac{1}{4}\exp(-4i\Delta^2 z-i\beta_0^2 z/2)\times\sqrt{3\beta_2}(\sqrt{\beta_2-\beta_0}+\sqrt{\beta_2+\beta_0})\operatorname{sech}^2\Delta x.$$


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